# Fuzzy Finite-State Automata Can Be Deterministically Encoded Into Recurrent Neural Networks *

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**Abstract**

There has been an increased interest in combining fuzzy systems with neural networks because fuzzy neural systems merge the advantages of both paradigms. On the one hand, parameters in fuzzy systems have clear physical meanings, and rule-based and linguistic information can be incorporated into adaptive fuzzy systems in a systematic way. On the other hand, there exist powerful algorithms for training various neural network models. However, most of the proposed combined architectures are only able to process static input-output relationships; they are not able to process temporal input sequences of arbitrary length. Fuzzy finite-state automata (FFAs) can model dynamical processes whose current state depends on the current input and previous states. Unlike in the case of deterministic finite-state automata (DFAs), FFAs are not in one particular state, rather each state is occupied to some degree defined by a membership function. Based on previous work on encoding DFAs in discrete-time, second-order recurrent neural networks, we propose an algorithm that constructs an augmented recurrent neural network that encodes a FFA and recognizes a given fuzzy regular language with arbitrary accuracy. We then empirically verify the encoding methodology by correct string recognition of randomly generated FFAs. In particular, we examined how the networks' performance varies as a function of synaptic weight strengths.

**Keywords:** Fuzzy systems, fuzzy neural networks, recurrent neural networks, knowledge representation, automata, languages, nonlinear systems.

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# 1 Introduction

## 1.1 Fuzzy Systems and Neural Networks

There has been an increased interest in combining artificial neural networks and fuzzy systems (see [4] for a collection of papers). Fuzzy logic [69] provides a mathematical foundation for approximate reasoning; fuzzy logic has proven very successful in a variety of applications [7, 9, 14, 23, 28, 29, 44, 67]. The parameters of adaptive fuzzy systems have clear physical meanings which facilitates the choice of their initial values. Furthermore, rule-based information can be incorporated into fuzzy systems in a systematic way.

Artificial neural networks simulate on a small scale the information processing mechanisms found in biological systems which are based on the cooperation of neurons that perform simple operations, and on their ability to learn from examples. Artificial neural networks have become valuable computational tools in their own right for tasks such as pattern recognition, control, and forecasting.

Fuzzy systems and multilayer perceptrons are computationally equivalent, i.e. they are both universal approximators [10, 62]. Recurrent neural networks have been shown to be computationally at least as powerful as Turing machines [55, 54]; whether or not these results also apply to recurrent fuzzy systems remains an open question. While the methodologies underlying fuzzy systems and neural networks are quite different, their functional forms are often similar. The development of learning algorithms for neural networks has been beneficial to the field of fuzzy systems which adopted some learning algorithms; there exist backpropagation training algorithms for fuzzy logic systems which are similar to the training algorithms for neural networks [20, 63].

## 1.2 Fuzzy Knowledge Representation in Neural Networks

In some cases, neural networks can be structured based on the principles of fuzzy logic [19, 46]. Neural network representations of fuzzy logic interpolation have also been used within the context of reinforcement learning [3].
Consider the following set of linguistic fuzzy rules:

IF ($x_1$ is $A_1$) AND ($x_2$ is $A_3$) THEN $C_1$.
IF (($x_1$ is $A_2$) AND ($x_2$ is $A_4$)) OR (($x_1$ is $A_1$) AND ($x_2$ is $A_3$)) THEN $C_2$.
IF ($x_1$ is $A_2$) AND ($x_2$ is $A_4$) THEN $C_3$.

where $A_i$ and $C_j$ are fuzzy sets and $x_i$ are linguistic input variables. A possible mapping of such rules into a feedforward neural network is shown in Fig. 1. These mappings are typical for control applications. The network has an input layer (layer 1) consisting of real-valued input variables $x_1$ and $x_2$ (e.g., linguistic variables), a fuzzification layer (layer 2) which maps input values $x_i$ to fuzzy sets $A_i$, an interpolation layer (layer 3) which computes the conjunction of all antecedent conditions in a rule (e.g., differential softmin operation), a layer which combines the contributions from all fuzzy control rules in the rule base (layer 4), and a defuzzification layer (layer 5) which computes the final output (e.g. mean of maximum method). Thus,

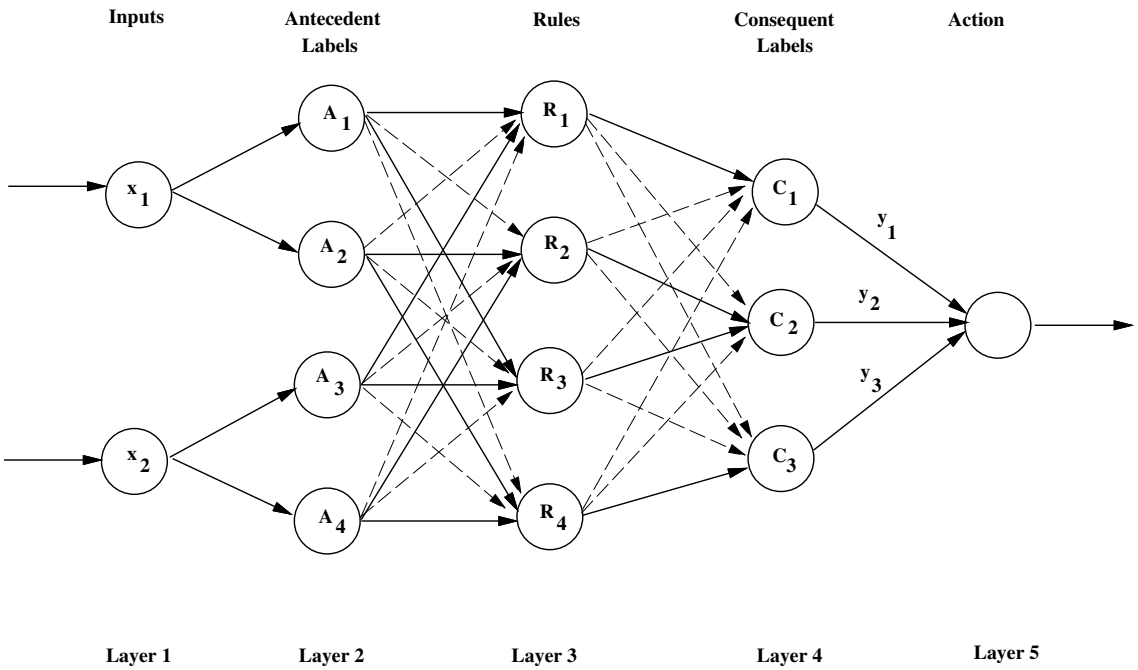


Figure 1: **Fuzzy Feedforward Neural Network:** An example of initializing a feedforward neural network with fuzzy linguistic rules neural network (solid directed connections). Additional weights may be added for knowledge refinement or revision through training (dashed directed connections). They may be useful for unlearning incorrect prior knowledge or learning unknown rule dependencies. Typically, radial basis functions of the form $e^{-(\frac{x-a}{b})^2}$ are used to represent fuzzy sets (layers 2 and 4); sigmoid discriminant functions of the form $\frac{1}{1+e^{-x}}$ are appropriate for computing the conjunction of antecedents of a rule (layer 3).

fuzzy neural networks play the role of *fuzzy logic interpolation engines.* [1] The rules are then tuned using a training algorithm for multilayer perceptrons. The extraction of fuzzy if-then-rules from trained multilayer perceptrons has also been investigated [19, 24, 38, 39].

There exist applications where the variables of linguistic rules are recursive, i.e., the rules are of the form

$$\text{IF } (x(t-1) \text{ is } \alpha) \text{ AND } (u(t-1) \text{ is } \beta) \text{ THEN } x(t) \text{ is } \gamma$$

where $u(t-1)$ and $x(t-1)$ represent previous input and state variables, respectively. The value of the state variable $x(t)$ depends on both the previous input $u(t-1)$ and the previous state $x(t-1)$. Clearly, feedforward neural networks generally do not have the computational capabilities to represent such recursive rules when the depth of the recursion is not known a priori. Recurrent neural networks are capable of representing recursive rules. (Feedforward neural networks with unit time delay feedback can represent recursive linguistic rules with recursive depth 1 if all state variables $x(t)$ are *observable*. However, if any of

[1] The term *fuzzy inference* is also often used to describe the function of a fuzzy neural network. We choose the term *fuzzy logic interpolation* in order to distinguish between the function of fuzzy neural networks and fuzzy logic inference where the objective is to obtain some properties of fuzzy sets $B_1, B_2, \ldots$ from properties of fuzzy sets $A_1, A_2, \ldots$ with the help of an inference scheme $A_1, A_2, \ldots \rightarrow B_1, B_2, \ldots$ which is governed by a set of rules [57, 58].

the state variables are *not observable* or have different recursive depths, then feedforward neural networks are inadequate.) Recurrent neural networks have the ability to store information over indefinite periods of time, can develop "hidden" states through learning, and are thus potentially useful for representing recursive linguistic rules.

A large class of problems where the current state depends on both the current input and the previous state can be modeled by finite-state automata or their equivalent grammars. It has been shown that recurrent neural networks can represent finite-state automata [5, 8, 11, 13, 16, 18, 32, 40, 47, 59, 60, 65, 70]. Thus, it is only natural to ask whether recurrent neural networks can also represent *fuzzy* finite-state automata (FFAs), and thus be used to implement recognizers of fuzzy regular grammars.

Fuzzy grammars have been found to be useful in a variety of applications such as in the analysis of X-rays [45], in digital circuit design [36], and in the design of intelligent human-computer interfaces [51]. The fundamentals of FFAs have been in discussed in [17, 50, 66] without presenting a systematic method for machine synthesis. Neural network implementations of fuzzy automata have been proposed in the literature [21, 22, 30, 61]. The synthesis method proposed in [21] uses digital design technology to implement fuzzy representations of states and outputs. In [61], the implementation of a Moore machine with fuzzy inputs and states is realized by training a feedforward network explicitly on the state transition table using a modified backpropagation algorithm. The fuzzification of inputs and states reduces the memory size that is required to implement the automaton with a microcontroller (e.g., antilock braking systems). In related work, an algorithm for implementing weighted regular languages in neural networks with probabilistic logic nodes was discussed [31]. A general synthesis method for synchronous fuzzy sequential circuits has been discussed in [64]. A synthesis method for a class of discrete-time neural networks with multilevel threshold neurons with applications to gray level image processing has been proposed in [53].

The combination of neural networks with fuzzy logic - so-called neuro-fuzzy systems - is enjoying increased popularity. However, it is very rare that these systems contain feedback in their structure [27]; thus, they are often inadequate for dealing with time-varying systems. On the one hand, the above mentioned applications demonstrate that fuzzy automata are gaining significance as synthesis tools for a variety of problems. On the other hand, recurrent neural networks are very promising candidates for modeling time-varying systems. They are particularly well-suited for problem domains where incomplete or contradictory prior knowledge is available. We have shown that recurrent networks can be initialized with that prior knowledge; the goal of training networks then becomes that of knowledge revision or refinement [42]. [2] Similarly, it would be useful to extend the role of recurrent networks as tools for knowledge revision and refinement to problem domains that can be modeled as FFAs and where prior (fuzzy) knowledge is available.

The purpose of this paper is to show that recurrent networks that can represent DFAs can be modified

[2] Maclin & Shavlik have demonstrated the power of combining DFAs and neural networks in the area of computational molecular biology (protein folding): They represented prior knowledge about protein structures as DFAs, initialized a neural network with that prior knowledge, and trained it on sample data. This approach outperformed the best known 'traditional' algorithm for protein folding [33, 34].

to accommodate FFAs. [3] The capability of *representing* FFAs can be viewed as a foundation for the problem of *learning* FFAs from example strings (if a network cannot represent FFAs, then it certainly cannot learn them either). Since DFAs are used in high-level VLSI design, and recurrent neural networks can be directly implemented in chips, this approach could be useful for applications where FFAs are used in conjunction with recurrent networks and VLSI [35, 52].

### 1.3 Outline of Paper

We briefly review deterministic, finite-state automata and their implementation in recurrent neural networks in Section 2. The extension of DFAs to FFAs is discussed in Section 3. In Section 4, we show how FFAs can be implemented in recurrent networks based on previous work on the encoding of DFAs [40]. In particular, our results show that FFAs can be encoded into recurrent networks such that a constructed network assigns membership grades to strings of arbitrary length with arbitrary accuracy. Simulation results in Section 5 validate our theoretical analysis. A summary and directions for future work in Section 6 conclude this paper.

## 2 Finite-State Automata and Recurrent Neural Networks

Here, we discuss the relationship between finite-state automata and recurrent neural networks for mapping fuzzy automata into recurrent networks. Details can be found in [40], and briefly with experimental verification in [43].

### 2.1 Deterministic Finite-State Automata

Regular languages represent the smallest class of formal languages in the Chomsky hierarchy [25]. Regular languages are generated by regular grammars.

**Definition 2.1** *A regular grammar $G$ is a quadruple $G =< S, N, T, P >$ where $S$ is the start symbol, $N$ and $T$ are non-terminal and terminal symbols, respectively, and $P$ are productions of the form $A \rightarrow a$ or $A \rightarrow aB$ where $A, B \epsilon N$ and $a \epsilon T$.*

The regular language generated by $G$ is denoted as $L(G)$.

An example of a regular language is the set of all strings over the alphabet $\Sigma = \{a, b\}$ where each string contains an even number of b's. The rules that generate all legal strings can be written in the form of production (or rewriting) rules:

$$S \rightarrow \epsilon \mid a \mid aS \mid bA$$
$$A \rightarrow aA \mid bS$$

The terminal and nonterminal symbols are 'a' and 'b', and $S$ and $A$, respectively ($S$ is also the start symbol of this grammar), and $\epsilon$ denotes the empty string. Each nonterminal symbol on the left-hand side of a rule can be replaced (or "rewritten") by any of the strings of terminal and nonterminal symbols on the right hand-side of

[3] Since the representation of FFAs in recurrent networks is based on results derived for the representation of DFAs, this paper highlights without proofs the relevant theoretical results; the details of the analysis and relevant proofs can be found in [40].

that rule. Thus, the string 'abba' would be generated by applying the rules $S \rightarrow aS \rightarrow abA \rightarrow abbS \rightarrow abba$. Other formal languages with fewer restrictions on the form of the right-hand side of a production rule exist (e.g., context-free grammars).

Associated with each regular language $L$ is a deterministic finite-state automaton (DFA) $M$ which is an acceptor for the language $L(G)$, i.e., $L(G) = L(M)$. DFA $M$ accepts only strings which are a member of the regular language $L(G)$.

**Definition 2.2** *A DFA $M$ is a 5-tuple $M =< \Sigma, Q, R, F, \delta >$ where $\Sigma = \{a_1, \ldots, a_m\}$ is the alphabet of the language $L$, $Q = \{q_1, \ldots, q_n\}$ is a set of states, $R \epsilon Q$ is the start state, $F \subseteq Q$ is a set of accepting states and $\delta : Q \times \Sigma \rightarrow Q$ defines state transitions in $M$.*

A string $s$ is accepted by the DFA $M$ and hence is a member of the regular language $L(M)$ if an accepting state in $F$ is reached after the string $s$ has been read by $M$. [4]

The DFA that accepts the strings containing an even number of b's is shown in Fig. 2.

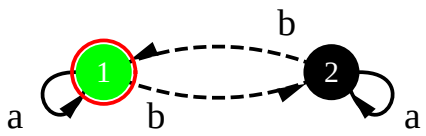


Figure 2: **DFA for Regular Language:** This DFA will accept all strings over the alphabet $\Sigma = \{a, b\}$ that contain an even numbers of b's. State 1 is the DFA's start and accepting state.

## 2.2 Network Construction

Various methods have been proposed for implementing DFAs in recurrent neural networks [1, 2, 6, 15, 16, 26, 32, 37, 41]. We use discrete-time, second-order recurrent neural networks with sigmoidal discriminant functions which update their current state according to the following equations:

$$S_i^{(t+1)} = h(\alpha_i(t)) = \frac{1}{1 + e^{-\alpha_i(t)}}, \qquad \alpha_i(t) = b_i + \sum_{j,k} W_{ijk} S_j^{(t)} I_k^{(t)}, \tag{1}$$

where $b_i$ is the bias associated with hidden recurrent state neurons $S_i$; $I_k$ denotes the input neuron for symbol $a_k$, $W_{ijk}$ is a second-order weight, $\alpha_i(t)$ is the total input to neuron $S_i$ at time $t$, and $h()$ is the sigmoidal discriminant function. The indices $i, j$ and $k$ run over the set of recurrent state neurons and the size of the language alphabet, respectively. The product $S_j^{(t)} I_k^{(t)}$ directly corresponds to the state transition $\delta(q_j, a_k) = q_i$. The network architecture is illustrated in Fig. 3.

[4] In analogy with programming languages, the regular grammar and DFA would correspond to the language definition and the parser for programs, respectively.

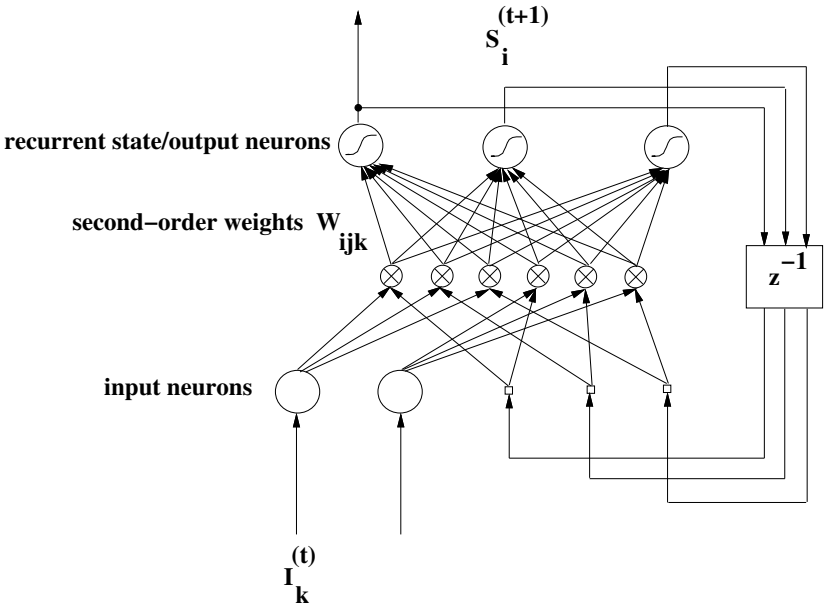


Figure 3: **Recurrent Network Architecture for Deterministic Finite-State Automata:** The recurrent state neurons are fully connected and implement the stable finite-state dynamics. One of the recurrent neurons also is the dedicated network output neuron (i.e., the neuron which by its output value classifies whether or not a given string is a member of a regular language).

We have proven that DFAs can be encoded in discrete-time, second-order recurrent neural networks with sigmoidal discriminant functions such that the DFA and constructed network accept and recognize the same regular language. Here, we give a brief description of the encoding algorithm; for more details, please see [40]. As previously mentioned, there are many encoding methods. The one described here is part of our particular proof process. The desired finite-state dynamics are encoded into a network by programming a small subset of all available weights to values $+H$ and $-H$. The goal is to maintain a nearly orthonormal internal DFA state representation: Only one state neuron $S_i$ that corresponds to the current DFA state has an output signal $\simeq 1$ (programmed weight $W_{ijk} = +H$); all other state neurons have output signals $\simeq 0$ (programmed weight $W_{jjk} = -H$) [5]. Weights which are not programmed to $+H$ or $-H$ are set to 0. Since we are using sigmoidal discriminant functions, programming the weights to $+H$ and $-H$ will cause the output of neurons to be high and low, respectively (see Fig. 4). The weight programming for the DFA shown in Fig. 2 is illustrated in the Table 1. There exist conditions that guarantee the stability of the internal DFA state representation for arbitrary string length; we will discuss these conditions in a later section. Similarly, the weights of a network's output neuron $S_0$ are programmed to $+H$ or $-H$ for correct string classification. The network construction algorithm *depends* on the nearly orthonormal internal DFA state representation for programming DFA state transitions. Instability of the internal representation leads to misclassification of strings.

The encoding algorithm leads to the following special form of the equation governing the network dynamics:

$$S_i^{(t+1)} = h(x, H) = \frac{1}{1 + e^{H(1-2x)/2}} \tag{2}$$

[5] Programming $W_{jjk} = -H$ is not mandatory, i.e. we could also set $W_{jjk} = 0$. However, our analysis applies to that particular encoding of DFA state transitions in recurrent networks. Other encoding are possible, but the analysis would have to be changed.

| $W_{0jk}$ | $W_{1jk}$ | $W_{2jk}$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 0 | 0 |
| 0 | 0 | 0 |
| 0 | 0 | 0 |
| $+H$ | $+H$ | $-H$ |
| $-H$ | $-H$ | $H$ |
| 0 | 0 | 0 |
| $-H$ | $-H$ | $+H$ |
| $+H$ | $+H$ | $-H$ |

Table 1: **Example of Weight Programming:** A recurrent network with 2 input neurons and 3 state neurons can implement the DFA shown in Fig. 2. The weight strength is the same for all programmed weights ($H$ or $-H$). Each column shows the weight values $W_{ijk}$ feeding from state neuron $S_j$ and input neuron $I_k$ to state neuron $S_i$. The index $j$ and $k$ run in alphabetical order from 0 to 2 and 1 to 2, respectively. Neuron 0, 1, and 2 represent the network output, DFA state 1, and DFA state 2, respectively.

where $x$ is the input to neuron $S_i$, and $H$ is the weight strength. The underlying assumption is that all neurons have an internal threshold $b_i = -H/2$, and receive inputs, weighted by the same strength $H$, from the same number of neurons. This represents a worst case which is used to prove network stability. The more neurons contribute to the input of any given neuron, the more likely the internal DFA state representation becomes unstable. Details on the worst case analysis can be found in [40, 43].

## 2.3 Network Stability

There exist only two kinds of signals in a constructed neural network that models a DFA: Recurrent state neurons have high output signals only when they correspond to the current DFA state; all other recurrent neurons have low output signals. The stability of the internal DFA representation depends on the value of the weight strength $H$ used to program the state transitions. If $H$ is chosen too small, then the internal DFA representation becomes unstable (i.e., state neurons $S_i$ which do not correspond to the current DFA state $q_i$ no longer have low output signals). Since our goal is to have a constructed neural network exactly model the state transitions of some DFA, the problem is to find a value $H_0$ such that for $H > H_0$, the internal DFA state representation remains stable for strings of *arbitrary* length. We achieve this goal by proving that there exist appropriate upper and lower bounds for low and high signals, respectively which, if sufficiently tight, guarantee the stability of low and high signals.

In the remainder of this section, we state results which establish that the stability of the internal representation can be achieved. The proofs of these results can be found in [40].

The terms *principal* and *residual inputs* will be useful for the following discussion:

**Definition 2.3** *Let $S_i$ be a neuron with low output signal $S_i^t$ and $S_j$ be a neuron with high output signal $S_j^t$. Furthermore, let $\{S_l\}$ and $\{S_{l'}\}$ be sets of neurons with output signals $\{S_l^t\}$ and $\{S_{l'}^t\}$, respectively, for which $W_{ilk} \neq 0$ and $W_{il'k} \neq 0$ for some input symbol $a_k$ and assume $S_j \in \{S_l\}$. Then, neurons $S_i$ and $S_j$ receive principal inputs of opposite signs from neuron $S_j$ and residual inputs from all other neurons $S_l$ and $S_{l'}$, respectively, when the network executes the state transition $\delta(q_j, a_k) = q_i$.*

Low signals are particularly sensitive to becoming corrupted because even though a neuron $S_l$ may not correspond to a DFA state $q_i$ when a network executes a DFA state transition $\delta(q_j, a_k) = q_i$, it may still receive residual inputs from other neurons if state transitions $\delta(q_x, a_k) = q_l$ exist. Over several time steps, neuron $S_l$ then computes an *iteration* of residual inputs which can ultimately lead to the situation where the low signal $S_l^t$ converges toward a high output. Similarly, high signals can become corrupted.

The following lemma establishes an upper bound for low signals in a constructed network:

**Lemma 2.1** *The low signals are bounded from above by the fixed point $\phi_f$ of the function $f$*

$$\begin{cases} f^0 = 0 \\ f^{t+1} = h(r \cdot f^t) \end{cases} \tag{3}$$

This lemma can be proven by induction on $t$. It is assumed that each neuron receives residual inputs from no more than $r$ other neurons. Such an upper bound $r$ obviously exists for any constructed network.

It is easy to see that the function to be iterated in equation (3) is $f(x, r) = \dfrac{1}{1 + e^{(H/2)(1-2rx)}}$. The graphs of the function $f(x, r)$ are shown in Fig. 4 for different values of the parameter $r$.

The function $f(x, r)$ has some desirable properties [40]:

**Lemma 2.2** *For any $H > 0$, the function $f(x, r)$ has at least one fixed point $\phi_f^0$.*

If $f(x, r)$ has only one fixed point for chosen $r$ and $H$, then the accumulation of residual inputs in neurons with initial output signals $S_l^0 = 0$ causes the iteration $f^t$ to converge toward a value $\simeq 1$. This can cause instability of the internal DFA state representation since this convergence happens simultaneously for all recurrent state neurons.

The following lemma states that more desirable fixed points may exist:

**Lemma 2.3** *There exists a value $H_0^-(r)$ such that for any $H > H_0^-(r)$, $f(x, r)$ has three fixed points $0 < \phi_f^- < \phi_f^0 < \phi_f^+ < 1$.*

The following lemma shows the convergence property of iterations of the function $f(x, r)$:

**Lemma 2.4** *If $f(x, r)$ has three fixed points $\phi_f^-, \phi_f^0$, and $\phi_f^+$, then*

$$\lim_{t \to \infty} f^t = \begin{cases} \phi_f^- & x_0 < \phi_f^0 \\ \phi_f^0 & x_0 = \phi_f^0 \\ \phi_f^+ & x_0 > \phi_f^0 \end{cases} \tag{4}$$

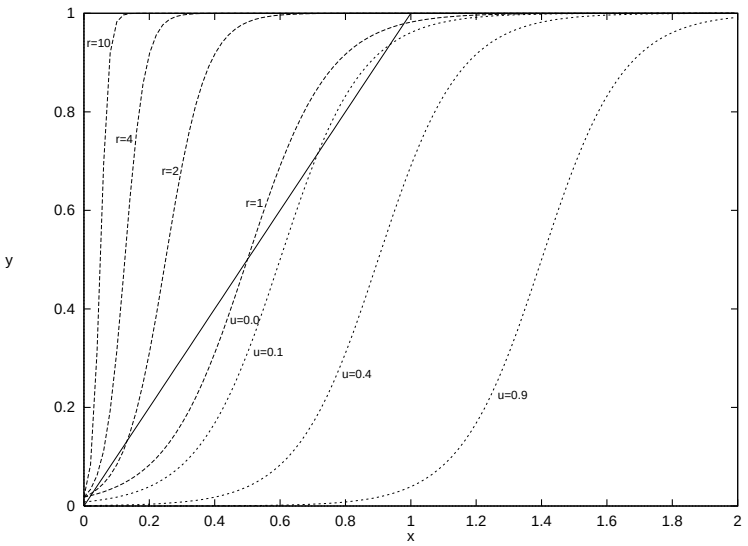


Figure 4: **Fixed Points of the Sigmoidal Discriminant Function**: Shown are the graphs of the function $f(x,r) = \frac{1}{1+e^{H(1-2rx)/2}}$ (dashed graphs) for $H = 8$ and $r = \{1,2,4,10\}$ and the function $p(x,u) = \frac{1}{1+e^{H(1-2(x-u))/2}}$ (dotted graphs) for $H = 8$ and $u = \{0.0, 0.1, 0.4, 0.9\}$. Their intersection with the function $y = x$ shows the existence and location of fixed points. In this example, $f(x,r)$ has three fixed points for $r = \{1,2\}$, but only one fixed point for $r = \{4,10\}$ and $p(x,u)$ has three fixed points for $u = \{0.0, 0.1\}$, but only one fixed point for $u = \{0.6, 0.9\}$.

The above lemma can be proven by defining an appropriate Lyapunov function $L$ and showing that $L$ has minima at $\phi_f^-$ and $\phi_f^+$ and that $f^t$ converges toward one of these minima. Notice that the fixed point $\phi_f^0$ is unstable.

Iteration towards a fixed point is only a necessary condition for the stability of low signals. In addition, convergence must occur monotonically:

**Lemma 2.5** *Let $f^0 = 0, f^1, f^2, \ldots$ denote the finite sequence computed by successive iteration of the function $f$. Then we have $f^0 < f^1 < \ldots < \phi_f^-$.*

With these properties, we can quantify the value $H_0^-(r)$ such that for any $H > H_0^-(r)$, $f(x,r)$ has three fixed points. The larger $r$, the larger $H$ must be chosen in order to guarantee the existence of three fixed points. If $H$ is not chosen sufficiently large, then $f^t$ converges to a unique fixed point $0.5 < \phi_f < 1$. The following lemma expresses a quantitative condition which guarantees the existence of three fixed points:

**Lemma 2.6** *The function $f(x,r) = \dfrac{1}{1+e^{(H/2)(1-2rx)}}$ has three fixed points $0 < \phi_f^- < \phi_f^0 < \phi_f^+ < 1$ if $H$ is chosen such that*

$$H > H_0^-(r) = \frac{2(1+(1-x)\ log(\frac{1-x}{x}))}{1-x}$$

*where $x$ satisfies the equation*

$$r = \frac{1}{2x(1+(1-x)\ log(\frac{1-x}{x}))}$$

The contour plots in Fig. 5 show the relationship between $H$ and $x$ for various values of $r$. If $H$ is chosen such that $H > H_0(r)$, then three fixed points exist; otherwise, only a single fixed point exists. The number and the location of fixed points depends on the values of $r$ and $H$. Thus, we write $\phi_f^-(r,H)$, $\phi_f^0(r,H)$, and $\phi_f^+(r,H)$, to denote the stable low, the unstable, and the stable high fixed point, respectively. We will use $\phi_f$ as a generic name for any fixed point of a function $f$.

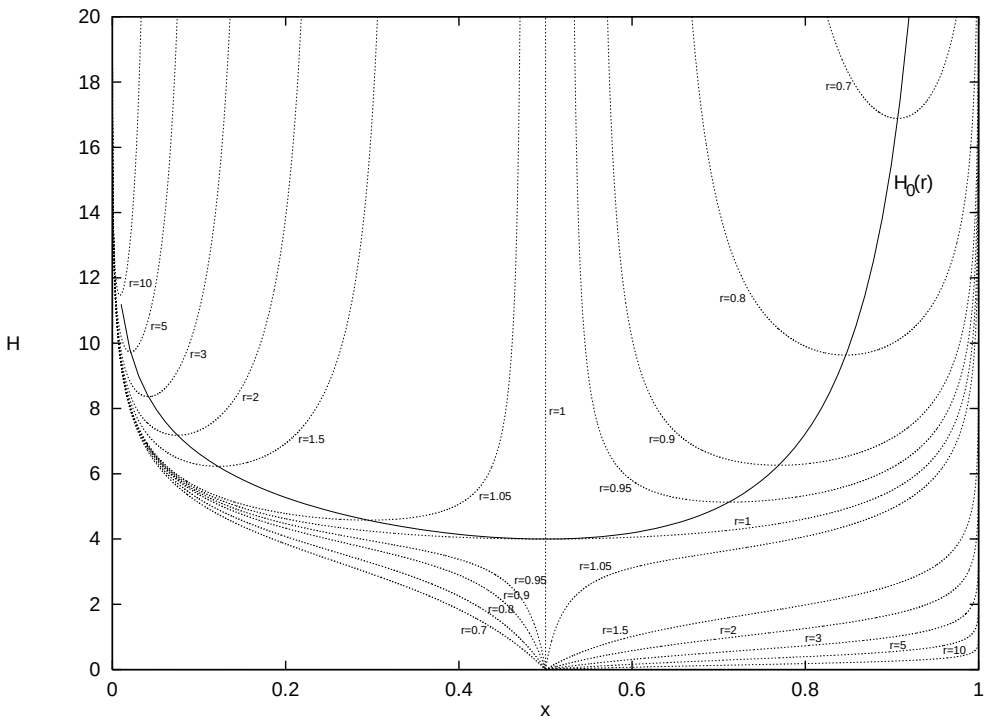


Figure 5: **Existence of Fixed Points**: The contour plots of the function $h(x,r) = x$ (dotted graphs) show the relationship between $H$ and $x$ for various values of $r$. If $H$ is chosen such that $H > H_0(r)$ (solid graph), then a line parallel to the x-axis intersects the surface satisfying $h(x,r) = x$ in three points which are the fixed points of $h(x,r)$.

Similarly, we can obtain a lower bound for high signals in a constructed network:

**Lemma 2.7** *Let $\phi_f$ denote the fixed point of the recursive function $f$, i.e. $\lim_{t\to\infty} f^t = \phi_f$. Then the high signals in a constructed recurrent neural network are bounded from below by the fixed point $\phi_g^+$ of the recursively defined function $g$*

$$\begin{cases} g^0 = 1 \\ g^{t+1} = h(g^t - \phi_f^-) \end{cases} \tag{5}$$

This can be proven by induction on $t$. Notice that we assume that the iteration $f^t$ converges toward $\phi_f^-$. The graphs of the function $g(x,u) = g(x,\phi_f^-)$ for some values of $u$ are shown in Fig. 4.

We can apply the same technique for finding conditions for the existence of fixed points of $g(x,u)$ as in the case of $f(x,r)$. In fact, the function that, when iterated, generates the sequence $g^0 = 1, g^1, g^2, \ldots$ is defined by

$$g(x,u) = g(x,\phi_f) = \frac{1}{1+e^{(H/2)(1-2(x-\phi_f^-))}} = \frac{1}{1+e^{(H'/2)(1-2r'x))}} \tag{6}$$

with

$$H' = H(1+2\phi_f), \;\; r' = \frac{1}{1+2\phi_f} \tag{7}$$

Since we can iteratively compute the value of $\phi_f$ for given parameters $H$ and $r$, we can repeat the original argument with $H'$ and $r'$ in place of $H$ and $r$ to find the conditions under which $g(r,x)$ has three fixed points. This results in the following lemma:

**Lemma 2.8** *The function* $g(x, \phi_f^-) = \dfrac{1}{1 + e^{(H/2)(1-2(x-\phi_f^-))}}$ *has three fixed points* $0 < \phi_g^- < \phi_g^0 < \phi_g^+ < 1$ *if* $H$ *is chosen such that*

$$H > H_0^+(r) = \frac{2(1 + (1-x)\ log(\frac{1-x}{x}))}{(1 + 2\phi_f^-)(1-x)}$$

*where* $x$ *satisfies the equation*

$$\frac{1}{1 + 2\phi_f^-} = \frac{1}{2x(1 + (1-x)\ log(\frac{1-x}{x}))}$$

We now define stability of recurrent networks constructed from DFAs:

**Definition 2.4** *An encoding of DFA states in a second-order recurrent neural network is called* <u>stable</u> *if all the low signals are less than* $\phi_f^0(r, H)$*, and all the high signals are greater than* $\phi_g^0(r, H)$*.*

The following result states conditions under which a constructed recurrent network implements a given DFA. They can be obtained by assuming a worst case where all neurons contribute to the instability of low and high signals:

**Theorem 2.1** *For some given DFA* $M$ *with* $n$ *states and* $m$ *input symbols, let* $r$ *denote the maximum number of transitions to any state over all input symbols of* $M$*. Then, a sparse recurrent neural network with* $n+1$ *sigmoidal state neurons and* $m$ *input neurons can be constructed from* $M$ *such that the internal state representation remains stable if the following three conditions are satisfied:*

$$(1) \quad \phi_f^-(r, H) < \frac{1}{r}(\ \frac{1}{2}\ + \frac{\phi_f^0(r, H)}{H})$$

$$(2) \quad \phi_g^+(r, H) > \frac{1}{2} + \phi_f^-(r, H) + \frac{\phi_g^0(r, H)}{H}$$

$$(3) \quad H > max(H_0^-(r), H_0^+(r))$$

*Furthermore, the constructed network has at most* $3mn$ *second-order weights with alphabet* $\Sigma_w = \{-H, 0, +H\}$*,* $n+1$ *biases with alphabet* $\Sigma_b = \{-H/2\}$*, and maximum fan-out* $3m$*.*

The number of weights and the maximum fan-out follow directly from the DFA encoding algorithm.

The above conditions implicitly put lower bounds on the magnitude of $H$ which guarantee stable finite-state dynamics for a network of given size. As such, they represent worst cases, i.e. the finite-state dynamics of a given neural network implementation may remain stable for smaller values of $H$ even for very large networks [40]. For any given DFA, the value of $H$ that satisfies the conditions of the above theorem must be determined numerically since neither the values of the fixed points nor the conditions for the existence of fixed points can be solved explicitly.

Since deterministic and fuzzy finite-state automata share a common underlying structure expressed in terms of state transitions, we will be able to use the result on the stability of the network dynamics for DFAs to implement fuzzy finite-state automata.

# 3 Fuzzy Finite-State Automata

Here we formally define fuzzy finite-state automata (FFAs), and describe their properties.

We begin by defining the class of fuzzy automata for which we develop a synthesis method for recurrent neural networks:

**Definition 3.1** *A fuzzy regular grammar $\widetilde{G}$ is a quadruple $\widetilde{G} =< S, N, T, P >$ where $S$ is the start symbol, $N$ and $T$ are non-terminal and terminal symbols, respectively, and $P$ are productions of the form $A \stackrel{\theta}{\rightarrow} a$ or $A \stackrel{\theta}{\rightarrow} aB$ where $A, B \,\epsilon\, N$, $a \,\epsilon\, T$ and $0 < \theta \leq 1$.*

Unlike in the case of DFAs where strings either belong or do not belong to some regular language, strings of a fuzzy language have graded membership:

**Definition 3.2** *Given a regular fuzzy grammar $\widetilde{G}$, the membership grade $\mu_{\widetilde{G}}(s)$ of a string $s$ in the regular language $L(\widetilde{G})$ is the maximum value of any derivation of $s$, where the value of a specific derivation of $s$ is equal to the minimum weight of the productions used:*

$$\mu_{\widetilde{G}(s)} = \mu_{\widetilde{G}}(S \stackrel{*}{\Rightarrow} s) = \max_{S \stackrel{*}{\Rightarrow} s} \min[\mu_{\widetilde{G}}(S \rightarrow \alpha_1), \mu_{\widetilde{G}}(\alpha_1 \rightarrow \alpha_2), \ldots, \mu_{\widetilde{G}}(\alpha_m \rightarrow s)]$$

This is akin to the definition of stochastic regular languages [48] where the min- and max-operators are replaced by the product- and sum-operators, respectively. Both fuzzy and stochastic regular languages are examples of weighted regular languages [49]. However, there are also distinct differences: In stochastic regular languages, the production rules are applied according to a probability distribution (i.e., the production weights are interpreted as probabilities). If, at any state of string generation, there exist several alternative rules, exactly one of them is applied; there exists no uncertainty about the generated string once the rule has been applied (note also that the production probabilities sum to 1). In fuzzy regular grammars, there is no question whether a production rule is applied; all applicable production rules are executed to some degree (note that the production weights do not need to sum to 1). This leaves some uncertainty or ambiguity about the generated string. Whether to choose stochastic or fuzzy regular grammars depends on the particular application.

**Definition 3.3** *A fuzzy finite-state automaton (FFA) $\widetilde{M}$ is a 6-tuple $\widetilde{M} =< \Sigma, Q, Z, \widetilde{R}, \delta, \omega >$ where $\Sigma$ and $Q$ are the same as in DFAs; $Z$ is a finite output alphabet, $\widetilde{R}$ is the fuzzy initial state, $\delta : \Sigma \times Q \times [0,1] \rightarrow Q$ is the fuzzy transition map and $\omega : Q \rightarrow Z$ is the output map.*

In this paper, we consider a restricted type of fuzzy automaton whose initial state is not fuzzy, and $\omega$ is a function from $F$ to $Z$, where $F$ is a non fuzzy subset of states, called *final states*. Any fuzzy automaton as described in definition 3.3 is equivalent to a restricted fuzzy automaton [12]. Notice that a FFA reduces to a conventional DFA by restricting the transition weights to 1. [6]

[6] Under the common definition of DFAs, we have $Z = \{0, 1\}$ (i.e., a string is either rejected or accepted). We extend that definition of DFAs by making the string membership function fuzzy: DFA $M =< \Sigma, Q, R, F, \delta, Z, \omega >$ where $Z$ is the finite set of all possible string memberships and $\omega : Q \rightarrow Z$ is a labeling defined for all DFA states.

As in the case of DFAs and regular grammars, there exist a correspondence between FFAs and fuzzy regular grammars [12]:

**Theorem 3.1** *For a given fuzzy grammar $\widetilde{G}$, there exists a fuzzy automaton $\widetilde{M}$ such that $L(\widetilde{G}) = L(\widetilde{M})$.*

Our goal is to use only continuous (sigmoidal and linear) discriminant functions for the neural network implementation of FFAs. The following results greatly simplifies the encoding of FFAs in recurrent networks with continuous discriminant functions:

**Theorem 3.2** *Given a regular fuzzy automaton $\widetilde{M}$, there exists a deterministic finite-state automaton $M$ with output alphabet $Z \subseteq \{\theta : \theta \text{ is a production weight}\} \cup \{0\}$ which computes the membership function $\mu : \Sigma^* \rightarrow [0,1]$ of the language $L(\widetilde{M})$.*

The algorithm, as given in [56], is shown in Fig. 6. An example of FFA-to-DFA transformation is shown in Fig. 7a [7]. An immediate consequence of this theorem is the following corollary:

**Corollary 3.1** *Given a regular fuzzy grammar $\widetilde{G}$, there exists an equivalent grammar $G$ in which productions have the form $A \stackrel{1.0}{\rightarrow} aB$ or $A \stackrel{\theta}{\rightarrow} a$.*

# 4 Recurrent Neural Network Architecture for Fuzzy Automata

We describe a method for encoding any fuzzy finite-state automata into a recurrent neural network. The result of Theorem 2.1 concerning the stability of the programmed network dynamics applies to finite-state automata whose states are *crisp* (i.e., the degree with which a state is the automaton's current state is either 0 or 1). On the other hand, FFAs can be in several states at any given time with different degrees of vagueness; vagueness is specified by a real number from the interval $[0,1]$.

Theorem 3.2 enables us to transform any FFA into a deterministic automaton which computes the same membership function $\mu : \Sigma^* \rightarrow [0,1]$. We just need to demonstrate how to implement the computation of $\mu$ with continuous discriminant functions.

For that purpose, we augment the network architecture used for encoding DFAs with additional weights which connect the recurrent state neurons to a linear output neuron (Fig. 8). The recurrent neurons implement the desired finite-state dynamics, i.e. transitions between crisp states (we showed in section 2.3 how to achieve stable finite-state dynamics for arbitrary string lengths). The linear output neuron computes the fuzzy membership of any given string. The weights connecting the recurrent state neurons with the linear output neuron are just the memberships assigned to the DFA states after the transformation of a FFA into an equivalent DFA. The algorithm for encoding FFAs in second-order recurrent neural networks is shown in

[7] This algorithm is an extension to the standard algorithm which transforms non-deterministic finite-state automata into equivalent deterministic finite-state automata [25]; unlike the standard transformation algorithm, we must distinguish accepting states with different fuzzy membership labels. Another method of decomposing a fuzzy grammar into qrisp grammars has been investigated by Zadeh using the concept of level set [68]

Input: Fuzzy regular grammar $\widetilde{G}$

Output: DFA $M$ such that $\forall s \in \Sigma^* : \mu_M(s) = \mu_{\widetilde{G}}(s)$.

1. Derive a FFA $\widetilde{M}$ from $\widetilde{G}$ using standard techniques.

   Each production of the form $A \stackrel{\theta}{\rightarrow} aB$ results in a state transition $\delta(A, a, \theta) = B$ where $B$ is a rejecting state. Productions of the form $A \stackrel{\theta}{\rightarrow} a$ denote the end of the derivation of a string. Thus, they result in a state transition $\delta(A, a, \theta) = X$ where $X$ is a sink, i.e., an accepting state. $S$ denotes the start state. The resulting FFA is shown in Fig. 7a.

   Example:

   $S \stackrel{0.3}{\rightarrow} aS \quad S \stackrel{0.5}{\rightarrow} aA \quad S \stackrel{0.7}{\rightarrow} aB \quad S \stackrel{0.3}{\rightarrow} bS \quad S \stackrel{0.2}{\rightarrow} bA \quad A \stackrel{0.5}{\rightarrow} b \quad B \stackrel{0.4}{\rightarrow} b$

2. Set $V = \{\theta \mid \theta > 0 \text{ is a production weight in } \widetilde{M}\}$

   Example:

   $V = \{0.2, 0.3, 0.4, 0.5, 0.7\}$

3. For each $\theta \in V$, obtain a regular expression $\widetilde{F}(\theta)$ describing those strings $s$ in the set

   $$L(\theta) = \{s \mid s \in \Sigma^* \text{ with } \mu_{\widetilde{M}}(s) \geq \theta\}.$$

   The threshold language $L(\theta)$ is known to be regular. $\widetilde{F}(\theta)$ can be obtained by ignoring all state transitions with production weights $< \theta$, and applying standard techniques for deriving regular expressions from the state diagrams.

   Example:

   $\theta = 0.2 \quad \widetilde{F}(0.2) = (a + b)^*(ab + bb)$

   $\theta = 0.3 \quad \widetilde{F}(0.3) = (a + b)^*ab$

   $\theta = 0.4 \quad \widetilde{F}(0.4) = ab$

   $\theta = 0.5 \quad \widetilde{F}(0.5) = ab$

   $\theta = 0.7 \quad \widetilde{F}(0.7) = \emptyset$

4. For each adjacent pair $\theta_1 > \theta_2 \in V$, beginning with lowest value, compute $F(\theta_2) = \widetilde{F}(\theta_2) \cap \overline{\widetilde{F}(\theta_1)}$

   This defines the set of strings

   $$\{s \mid s \in \Sigma^*, s \in L(\theta_2), s \in \overline{L(\theta_1)}\} = \{s \mid s \in \Sigma^*, \mu(s) \geq \theta_2, \mu(s) < \theta_1\} = \{s \mid s \in \Sigma^*, \mu(s) = \theta_2\}$$

   Example:

   $F(0.2) = (a + b)^*bb$

   $F(0.3) = (a + b)(a + b)^*ab$

   $F(0.4) = \emptyset$

   $F(0.5) = ab$

   $F(0.7) = \emptyset$

5. Derive a DFA from the regular expressions $\bigcup_{\theta \in V} F(\theta)$.

   This union defines a non-deterministic finite-state automata (NDFA) which can be transformed into an equivalent DFA using standard algorithms [25].

Figure 6: **Algorithm for Transformation of a FFA into an Equivalent DFA**

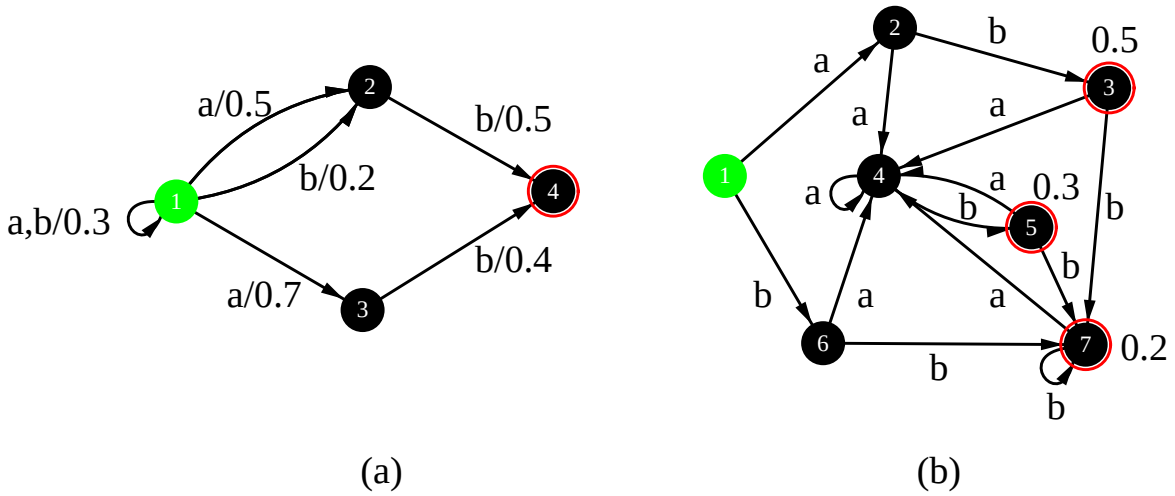


Figure 7: **Example of a Transformation of a Specific FFA into its Corresponding DFA**: (a) A fuzzy finite-state automaton with weighted state transitions. State 1 is the automaton's start state; accepting states are drawn with double circles. Only paths that can lead to an accepting state are shown (transitions to garbage states are not shown explicitly). A transition from state $q_j$ to $q_i$ on input symbol $a_k$ with weight $\theta$ is represented as a directed arc from $q_j$ to $q_i$ labeled $a_k/\theta$. (b) corresponding deterministic finite-state automaton which computes the membership function strings. The accepting states are labeled with the degree of membership. Notice that all transitions in the DFA have weight 1.

Fig. 9.

Recall the upper and lower bounds on the low and high signals, respectively (Lemmas 2.1 and 2.8). Let $\mu_i$ denote the graded memberships assigned to DFA states $q_i$. In the worst case, the network computes for a given string $s$ the fuzzy membership function

$$\mu_{RNN}(s) = \mu_i \, \phi_g^+ + (n_{acc} - 1) \, \phi_f^-$$

where $n_{acc}$ is the number of DFA states with $\mu_i > 0$.

Since $\phi_f^-$ and $\phi_g^+$ converge toward 0 and 1, respectively for increasing values of $H$, $\mu_{RNN}$ converges toward $\mu_i$. Notice that $|\mu_{RNN} - \mu_i|$ can be made arbitrarily small by increasing $H$.

## 5 Simulation Results

In order to empirically test our encoding methodology, we examined how well strings from a randomly generated FFAs are classified by a recurrent neural network in which the FFA is encoded. We randomly generated deterministic acceptors for fuzzy regular languages over the alphabet $\{0, 1\}$ with 100 states as

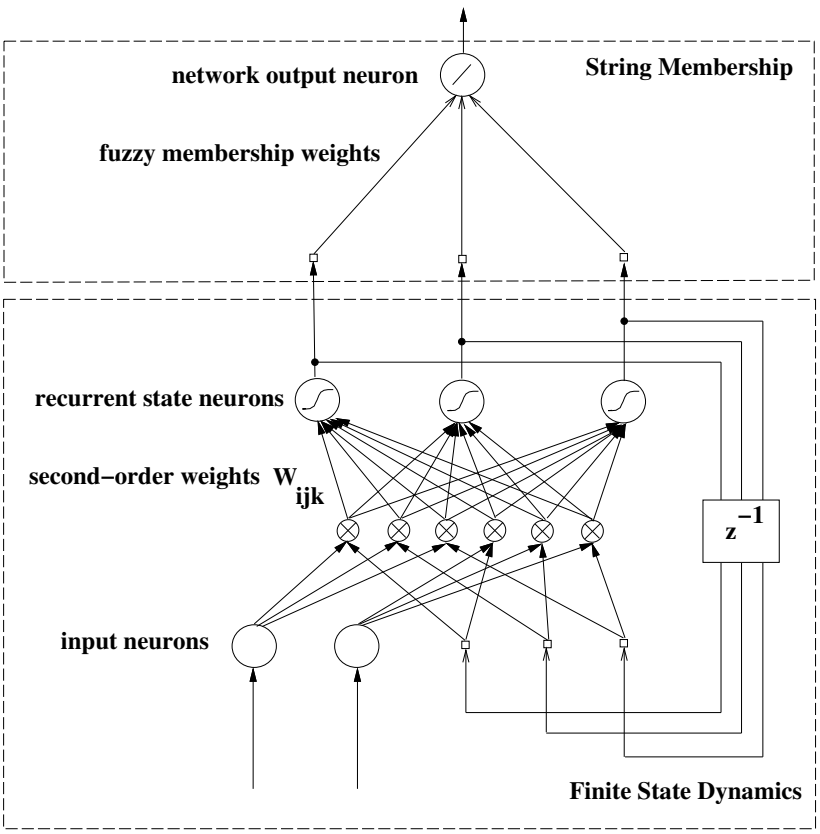


Figure 8: **Recurrent Network Architecture for Fuzzy Finite-State Automata**: The architecture consists of two parts: Recurrent state neurons encode the state transitions of the deterministic acceptor. These recurrent state neurons are connected to a linear output neuron which computes string membership.

follows: For each DFA state, we randomly generated a transition for each of the two input symbols to another state. Each accepting DFA state $q_i$ was assigned a membership $0 < \mu_i \leq 1$; for all non-accepting states $q_j$, we set $\mu_j = 0$. We encoded these acceptors into recurrent networks with 100 recurrent state neurons, two input neurons (one for each of the two input symbols 0 and 1), and one linear output neuron. We measured their performance on 100 randomly generated strings of fixed length 100, and on 100 randomly generated strings of varying length up to 100 - the majority of strings were much shorter - whose membership was determined from their deterministic acceptors. The graphs in Fig. 10 show the average absolute error of the network output as a function of the weight strength $H$ used to encode the finite-state dynamics for DFAs where 1%, 5%, 20%, 30%, 50% and 100% of all states had labels $0 < \mu_i \leq 1$. We observe that the error exponentially decreases with increasing hint strength $H$ (i.e., the average output error can be made arbitrarily small), and that the results for the two separate experiments are almost identical. This observation indicates that the DFA size has no significant impact on the network performance. The network performance depends on the stability of the internal representation of the finite-state dynamics. The value of $H$ for which the dynamics of all DFAs used in these experiments remains stable for strings of arbitrary length is approximately $H \simeq 9.8$. This value has to be chosen by a numerical solution of the conditions (1)-(3) of Theorem 2.1. When the representation becomes unstable because the weight strength $H$ has been chosen too small, then that instability occurs for very short strings (typically less than 5 iterations). Thus, the mixture of strings of varying length vs. long strings of fixed length has no significant impact on the network performance.

Input: FFA $\widetilde{M} =< \Sigma, Q, R, Z, \delta, \omega >$ with $\Sigma = \{a_1, a_2, \ldots, a_K\}$, $Q = \{q_1, q_2, \ldots, q_N\}$, $R = \{q_1\}$ is the crisp start state, $F \subseteq Q$ is the set of accepting states, $Z$ is the output alphabet, $\delta : \Sigma \times Q \to Q$ are the state transitions, $\omega : Q \to Z$ is the output map.
Output: Second-order recurrent neural network with $L(RNN) = L(FFA)$, i.e., the recurrent network assigns with arbitrary precision the same membership to all strings as the fuzzy automaton. The network is unique up to labeling of neurons.

1. Algorithm Initialization:

   Transform $\widetilde{M}$ into a unique deterministic acceptor $M$ with $N' > N$ states that computes the string membership $\mu_G$ for arbitrary strings. Let $M$ be in state $q_i$ after some string $s$ has been read; then the label $0 < \mu_i \leq 1$ associated with state $q_i$ indicates the membership assigned to $s$ by $M$.
   choose $N'$ neurons with sigmoidal discriminant function $g(x) = \frac{1}{1+e^{-x}}$ and one nonrecurrent output neuron with linear discriminant function.
   For each $a_k \in \Sigma$ construct an input vector $(0, \ldots, 0, I_{k-1}, 1, I_{k+1}, 0, \ldots, 0)$ of length $K$.
   Numerically determine weight strength $H$ which satisfies the conditions of Theorem 2.1.
2. Network Construction:

   for $i = 1 \ldots N'$
   $b_i = -H/2$;
   for $j = 1 \ldots N'$
   for $k = 1 \ldots K$
   $W_{ijk} = +H$ if $\delta(q_j, a_k) = q_i$; $W_{jjk} = -H$ if $\delta(q_j, a_k) \neq q_j$;
   $W_{0ik} = \mu_i$;
3. Network Dynamics:

   $$S_i^{t+1} = g(b_i + \sum_{j,k,j>0} W_{ijk}\, S_j^t\, I_k^t) \quad (i > 0)$$
   $$S_0^{t+1} = \sum_{j>0} \mu_j S_j^{t+1}$$
4. Network Initialization:

   Before reading a new string, set the initial network state to
   $\mathbf{S^0} = (S_0^0, S_1^0, \ldots, S_{N'}^0) = (S_0^0, 1, 0, \ldots, 0)$; the value of $S_0^0$ is computed as $S_0^0 = \mu_1$ (see network dynamics). .
5. Network Performance:

   The output of $S_0^f$ after reading a string $s$ of length $f$ is the string membership $\mu_G(s)$.

6. Network Complexity:

   number of neurons: N'+1; number of weights: O(KN'); maximum fan-out: 3K
   weight alphabet $\Sigma_W = \{-H, -H/2, 0, H, \mu_1, \ldots, \mu_{N'}\}$

Figure 9: **Algorithm for Encoding Arbitrary FFAs in Second-Order Recurrent Neural Networks**

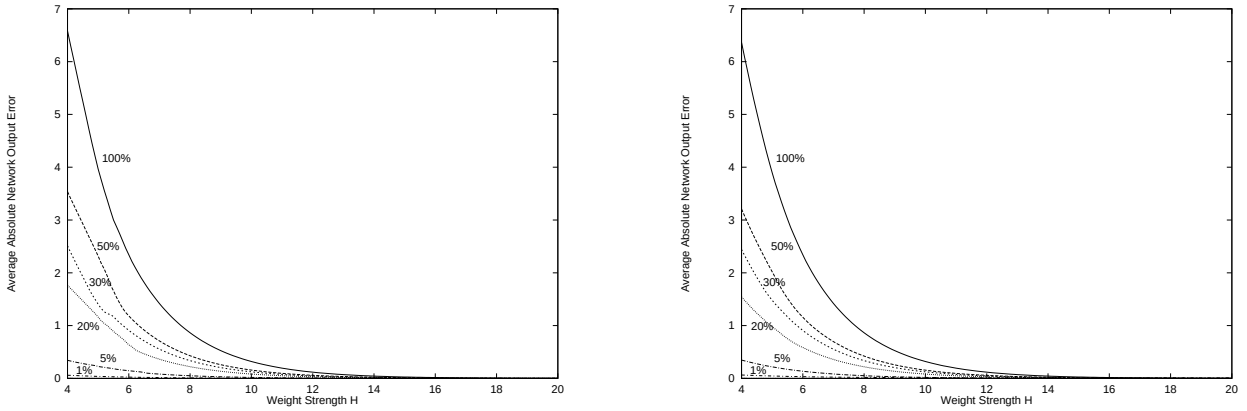


Figure 10: **Neural Network Performance**: The graphs show average absolute error of the neural network output when tested on (a) 100 randomly generated strings of fixed length 100 and (b) on 100 randomly generated strings of length up to 100 as a function of the weight strength $H$ used to encode the finite-state dynamics of randomly generated DFAs with 100 states. The percentages of DFA states with $\mu_i > 0$ were 1%, 5%, 20%, 30%, 50% and 100% respectively, of all DFA states.

# 6 Conclusions

We have proposed a method for representing fuzzy finite-state automata (FFAs) in recurrent neural networks with continuous discriminant functions. Based on a previous result on encoding stable representations of finite-state dynamics in recurrent networks, we have shown how FFAs can be encoded in recurrent networks that compute string membership functions with arbitrary accuracy. The method uses an algorithm which transforms FFAs into equivalent DFAs which compute fuzzy string membership. The fuzzy FFA states are transformed into crisp DFA states. A membership label $\mu_i$ with $0 < \mu_i \leq 1$ is associated with each accepting DFA state; nonaccepting DFA states have label $\mu_i = 0$. The membership of a string is equal to the membership label of the last visited DFA state.

A recurrent neural network is constructed from the original architecture used for DFA encodings by connecting the recurrent state neurons to a linear output neuron. The weights of these connections are set to the value of the membership labels of the DFA states. The accuracy of the computation of the string membership function depends on the network size, the number of DFA states which membership label $\mu_i > 0$, and the weight strength $H$ used to encode the finite-state dynamics in the recurrent network. The larger $H$ is chosen, the more accurate the network computes membership functions.

The significance of the results presented here is that deterministic recurrent neural networks are computational models rich enough to represent FFAs, and that the results on DFA representations carry over to the fuzzy domain. An interesting question is whether representations of FFAs can be *learned* through training on example strings, and how weighted production rules are represented in deterministic trained networks. We believe that a network that is trained on example strings will develop a *deterministic* internal representation of the FFA that generated the labels of the example strings, similar to our FFA encoding. This could be verified by applying DFA extraction methods to networks that were trained on strings with

fuzzy membership.

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